LENGTH OF PORTION OF EULER LINE INSIDE A TRIANGLE

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Abstract – The Euler line which was discovered in 1763 by Swiss mathematician Leonhard Euler, is a line that goes through the orthocenter, the centroid and the circumcenter of a non-equilateral triangle. Moreover, the distance between the orthocenter and the centroid is always double the distance between the centroid and circumcenter. This paper aims at finding the length of segment of Euler line that lies inside of a non-right scalene triangle if the length of its three sides are given. This paper aims at deriving two approaches to find the length of portion of Euler line that lies inside of a scalene non-right triangle. As a result of the approaches, this papers contributes by providing formulas that can be used for any scalene non-right triangle.

Key Words **–** Triangle, Euler line, Formula to calculate length

INTRODUCTION

Around 250 years ago, Swiss mathematician and physicist, Leonhard Euler, showed that in any non-equilateral triangle, the orthocenter, circumcenter, and centroid are collinear and the line which passes from these three points is called Euler line of the triangle.

Euler line in different types of triangles

There is no Euler line in an equilateral triangle as the orthocenter, circumcenter, and centroid of the triangle coincide. The Euler line of an isosceles triangle coincides with axis of symmetry of that triangle. In a right triangle, the Euler line coincides with the median to the hypotenuse. For an acute angled triangle, centroid, circumcenter and orthocenter lie in the interior of the triangle. For an obtuse angled triangle, the centroid lies in the interior while the orthocenter and circumcenter lie in the exterior of the triangle.

Application of Euler line

An important application of Euler line is that information about any one of the centroid, orthocenter, and circumcenter can be derived from the information of other two in a triangle.

Motivation for this paper

It was when I was studying Law of Cosines that it occurred to me that if an angle of a triangle can be expressed in terms of three sides of a triangle, then one can also express the length of portion of Euler line inside a triangle in terms of three sides of a triangle.

I did further study and came up with formulas to derive the length of portion of Euler line that lies inside of a non-equilateral triangle using known results.

Scope of this paper

Formulas for finding the length of portion of Euler line that lies inside of different types of triangles are given in the following table.

Type of triangle based on sides	Type of triangle based on angles	Formula for length of portion of Euler line that lies inside of a triangle
Equilateral	Acute	Euler line does not exist
	Right	Right equilateral triangle does not exist
	Obtuse	Obtuse equilateral triangle does not exist
Isosceles	Acute	Length of median to base*
	Right	Length of median to hypotenuse#
	Obtuse	Length of median to base*

Scalene	Acute	Will be derived in this paper
	Right	Length of median to hypotenuse#
	Obtuse	Will be derived in this paper

*In isosceles triangle, length of median to base (unequal side of isosceles triangle) is the length of portion of Euler line that lies inside of a triangle.

#Length of median to hypotenus, whose length is half of hypotenus, is the length of portion of Euler line that lies inside of a triangle.

In this paper, we will cover derivation of formulas for following two types of triangle –

I. For acute scalene triangle
II. For obtuse scalene triangle

Notations used in this paper

Let us consider a scalene Δ ABC in which e is the Euler line which intersects two sides of a triangle at points P and Q. AN, BE and CL are altitudes on BC, AC and AB respectively which intersects each other at orthocenter, H. AM is median to BC and CS is median to AB which intersects each other at centroid, G.

Let BC=a, AC=b, AB=c, $\angle A$=A, $\angle B$=B and $\angle C$=C.

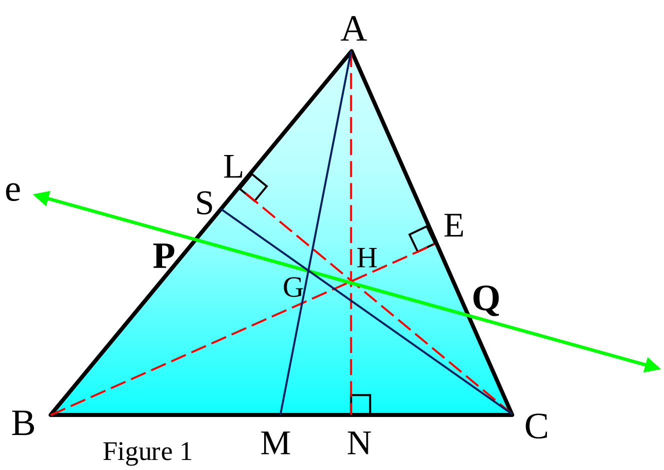

Figure 1

In order to derive the formulas, we will use the following known results:

i. Apollonius's Theorem [1]:
$$AB^2 + AC^2 = 2(AM^2 + BM^2)$$
ii. Centroid Theorem [2]:
$$AG = \frac{2}{3}AM$$
iii. Law of Sines [3]:
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
iv. Law of Cosines [4]:
$$c^2 = a^2 + b^2 - 2ab\cos C$$

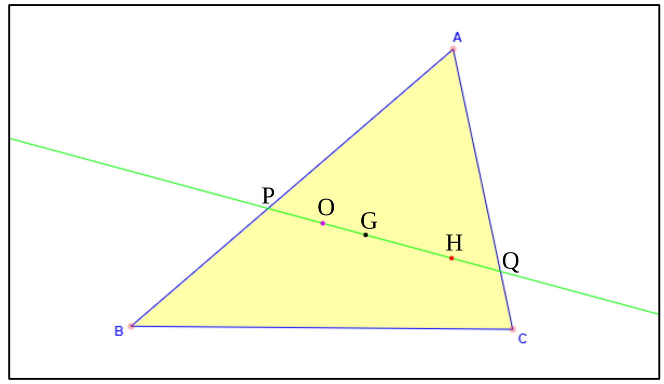

Figure 2: Euler line in acute triangle

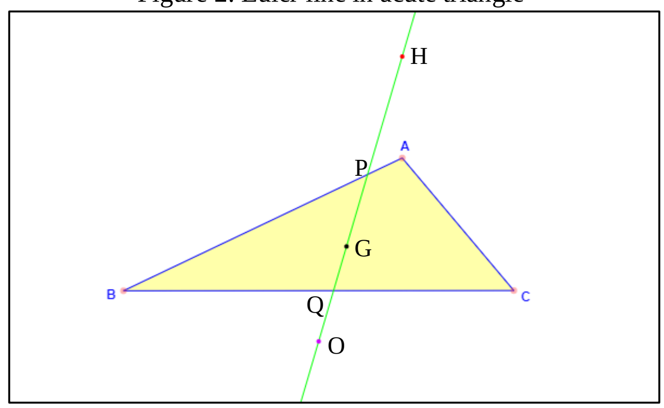

Figure 3: Euler line in obtuse triangle

In figure 2 and 3, $\overleftrightarrow{PQ}$ is the Euler line of triangle, which passes through orthocenter, H, centroid, G, and circumcenter, O.

These points have a unique property, $HG = 2GO$ [6].

I. DERIVATION OF FORMULA FOR ACUTE SCALENE TRIANGLE

Let Δ ABC be acute triangle.

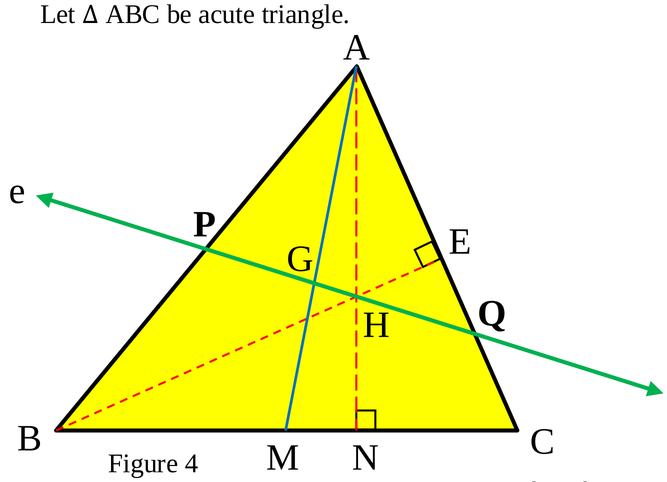

Figure 4

By Law of Cosines,

$$\angle A = cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$$
$$\angle B = cos^{-1}\left(\frac{a^2+c^2-b^2}{2ac}\right)$$
$$\angle C = cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

By Apollonius's Theorem,

$$AM = \tfrac{1}{2}\sqrt{2c^2+2b^2-a^2} \quad (1)$$

By Centroid Theorem,

$$AG = \frac{2}{3}AM$$
$$\therefore AG = \tfrac{1}{3}\sqrt{2c^2+2b^2-a^2} \quad (2)$$

For ΔAMC, by Law of Cosines,

$$\angle \mathrm{MAC} = \cos^{-1}\left(\frac{\mathrm{AM}^2+\mathrm{b}^2-\left(\frac{\mathrm{a}}{2}\right)^2}{2\cdot \mathrm{AM}\cdot \mathrm{b}}\right) \quad (3)$$

In eqn. (3), ∠MAC is an acute angle.

Also, $\angle NAC = 90° - \angle C$

$$\angle MAN = \angle GAH = \angle MAC - \angle NAC \quad (4)$$

$\sin B = \frac{AN}{c}$

$$\therefore AN = AH + HN = c\cdot\sin B \quad (5)$$

For ΔHAE, $\cos\angle HAE = \frac{AE}{AH}$

$$\therefore AH = \frac{AE}{\cos\angle HAE} \quad (5.1)$$

For ΔABE, $\cos A = \frac{AE}{c}$

$$\therefore AE = c\cdot\cos A \quad (5.2)$$

From (5.1) and (5.2),

$$AH = \frac{c\cdot\cos A}{\cos\angle HAE} \quad (5.3)$$

For ΔANC, $\angle NAC = \angle HAE = 90° - \angle C$

$$\therefore \cos\angle HAE = \cos(90° - \angle C)$$
$$\therefore \cos\angle HAE = \sin C$$

Therefore, from (5.3),

$$AH = \frac{c\cdot\cos A}{\sin \mathrm{C}} \quad (5.4)$$

For ΔHBN, $\tan\angle HBN = \frac{HN}{BN}$

$$\therefore HN = BN\cdot\tan\angle HBN \quad (5.5)$$

For ΔBEC, $\angle HBN = 90° - \angle C$

$$\therefore \tan\angle HBN = \cot C$$
$$\therefore \tan\angle HBN = \frac{\cos C}{\sin C}$$

Therefore, from (5.5),

$$HN = BN\cdot\frac{\cos C}{\sin C} \quad (5.6)$$

For ΔABN, $\cos B = \frac{BN}{c}$

$$\therefore BN = c\cdot\cos B$$

Therefore, from (5.6),

$$HN = c\cdot\cos B\cdot\frac{\cos C}{\sin C} \quad (5.7)$$

From (5.4) and (5.7),

$$\frac{AH}{HN} = \frac{\frac{c\cdot\cos A}{\sin C}}{c\cdot\cos B\cdot\frac{\cos C}{\sin C}}$$
$$\therefore \frac{AH}{HN} = \frac{\cos A}{\cos B\cdot\cos C}$$

$$\therefore \frac{AH}{\cos A} = \frac{HN}{\cos B\cdot\cos C} = x$$
$$\therefore AH = \cos A\cdot x \text{ and } HN = \cos B\cdot\cos C\cdot x \quad (5.8)$$
$$AH + HN = x(\cos A + \cos B\cdot\cos C)$$

But $AH + HN = c\cdot\sin B$ From (5)

So, $x(\cos A + \cos B\cdot\cos C) = c\cdot\sin B$

$$\therefore x = \frac{c\cdot\sin B}{\cos A + \cos B\cdot\cos C}$$

But $AH = \cos A\cdot x$ From (5.8)

$$\therefore AH = \frac{c\cdot\sin B\cdot\cos A}{\cos A + \cos B\cdot\cos C} \quad (6)$$

For ΔGAH, by Law of Cosines,

$$GH = \sqrt{AG^2 + AH^2 - 2\cdot \mathrm{AG}\cdot\mathrm{AH}\cdot\cos\angle\mathrm{GAH}} \quad (7)$$

$$\angle AGH = cos^{-1}\left(\frac{AG^2+GH^2-AH^2}{2\cdot AG\cdot GH}\right) \quad (8)$$

For ΔAGQ, $\angle GAQ = \angle MAC$, $\angle AGQ = \angle AGH$

$$\angle AQG = 180° - \angle GAQ - \angle AGQ \quad (9)$$

By Law of Sines,

$$\frac{GQ}{\sin\angle GAQ} = \frac{AG}{\sin\angle AQG}$$
$$\therefore \boldsymbol{GQ} = \frac{AG\cdot\sin\angle GAQ}{\sin\angle AQG} \quad (10)$$

For ΔAGP, $\angle AGP = 180° - \angle AGQ$

$$\angle PAG = \angle A - \angle GAQ \quad (11)$$

$$\angle APG = 180° - \angle AGP - \angle PAG$$
$$= 180° - (180° - AGQ) - (\angle A - \angle GAQ)$$
$$\therefore \angle APG = \angle AGQ + \angle GAQ - \angle A \quad (12)$$

For ΔAGP, by Law of Sines,

$$\frac{PG}{\sin\angle PAG} = \frac{AG}{\sin\angle APG}$$

$$\therefore \boldsymbol{PG} = \frac{AG\cdot\sin\angle PAG}{\sin\angle APG} \quad (13)$$

From (10) and (13),

$$PG + GQ = \frac{AG\cdot\sin\angle PAG}{\sin\angle APG} + \frac{AG\cdot\sin\angle GAQ}{\sin\angle AQG}$$

$$\therefore \mathrm{PQ} = \mathrm{AG} \cdot \left[\frac{\sin(\angle\mathrm{A} - \angle\mathrm{GAQ})}{\sin(\angle\mathrm{AGQ} + \angle\mathrm{GAQ} - \angle\mathrm{A})} + \frac{\sin(\angle\mathrm{GAQ})}{\sin(180° - \angle\mathrm{GAQ} - \angle\mathrm{AGQ})}\right]$$

Let AM=m, GH=d and AH=n. Using eqns. (1) to (13),

$$\boldsymbol{PQ} = \frac{2}{3}m \cdot \left[\frac{\sin\left(\angle A - \left\{\cos^{-1}\left(\frac{m^2+b^2-\left(\frac{a}{2}\right)^2}{2\cdot m\cdot b}\right)\right\}\right)}{\sin\left(\cos^{-1}\left(\frac{\left(\frac{2}{3}m\right)^2+d^2-n^2}{\frac{4}{3}m\cdot d}\right) + \cos^{-1}\left(\frac{m^2+b^2-\left(\frac{a}{2}\right)^2}{2\cdot m\cdot b}\right) - \angle A\right)} + \frac{\sin\left(\cos^{-1}\left(\frac{m^2+b^2-\left(\frac{a}{2}\right)^2}{2\cdot m\cdot b}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{m^2+b^2-\left(\frac{a}{2}\right)^2}{2\cdot m\cdot b}\right) + \cos^{-1}\left(\frac{\left(\frac{2}{3}m\right)^2+d^2-n^2}{\frac{4}{3}m\cdot d}\right)\right)}\right]$$

This formula is derived with respect to Figure 4.
If three sides of an acute scalene triangle are given, using eqns. (1) to (13) and the final formula, we can find the length of portion of Euler line inside any acute scalene triangle.

Now we will derive formula for obtuse scalene triangle.

II. DERIVATION OF FORMULA FOR OBTUSE SCALENE TRIANGLE

Let Δ ABC be obtuse triangle.

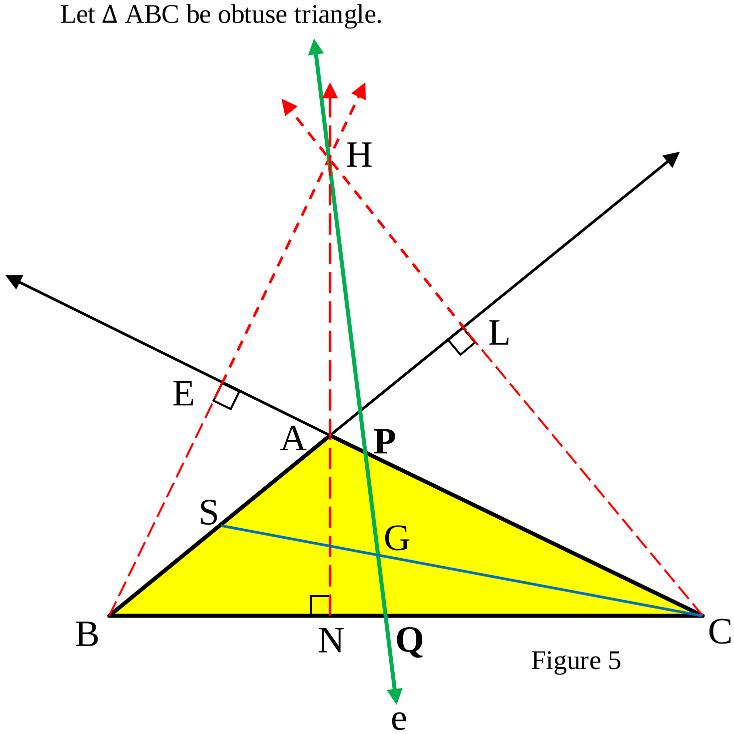

Figure 5

By Law of Cosines,

$$\angle A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$\angle B = \cos^{-1}\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$\angle C = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

By Apollonius's Theorem,

$$CS = \frac{1}{2}\sqrt{2b^2+2a^2-c^2} \tag{14}$$

By Centroid Theorem,

$$CG = \frac{2}{3}CS$$

$$\therefore CG = \frac{1}{3}\sqrt{2b^2+2a^2-c^2} \tag{15}$$

For ΔASC, by Law of Cosines,

$$\angle PCG = \angle ACS = \cos^{-1}\left(\frac{b^2+CS^2-\left(\frac{c}{2}\right)^2}{2\cdot b\cdot CS}\right) \tag{16}$$

where $\angle ACS$ is an acute angle.

For ΔALC,

$$\angle ACL = \angle A - 90° \tag{17}$$

$$\angle HCG = \angle ACL + \angle ACS \tag{18}$$

For ΔALC,

$$\cos\angle ACL = \frac{LC}{b}$$

From (17),

$$LC = b\cdot\cos(\angle A - 90°) \tag{19}$$

Again for ΔALC,

$$\sin\angle ACL = \frac{AL}{b}$$

Again from (17),

$$AL = b\cdot\sin(\angle A - 90°) \tag{20}$$

For ΔHAL,

$$\angle HAL = \angle BAN = 90° - \angle B \tag{21}$$

Also,

$$\tan\angle HAL = \frac{HL}{AL}$$

$$\therefore HL = AL\cdot\tan\angle HAL$$

From (20) and (21),

$$HL = b\cdot\sin(\angle A - 90°)\cdot\tan(90° - \angle B) \tag{22}$$

For ΔHGC, $\qquad HC = HL + LC$

From (19) and (22),

$$HC = [b \cdot \sin(\angle A - 90°) \cdot \tan(90° - \angle B)] + [b \cdot \cos(\angle A - 90°)] \quad (23)$$

By Law of Cosines,

$$HG = \sqrt{HC^2 + CG^2 - 2 \cdot HC \cdot CG \cdot \cos\angle HCG} \quad (24)$$

By Law of Sines,

$$\frac{\sin\angle HGC}{HC} = \frac{\sin\angle HCG}{HG}$$

$$\therefore \sin\angle HGC = \frac{\mathrm{HC} \cdot \sin\angle HCG}{HG}$$

Since $\angle HGC$ is obtuse,

$$\angle HGC = \angle PGC = 180° - \sin^{-1}\left(\frac{\mathrm{HC} \cdot \sin\angle HCG}{HG}\right)$$

For ΔPGC,

$$\angle GPC = 180° - \angle PCG - \angle PGC \quad (26)$$

By Law of Sines,

$$\frac{PG}{\sin\angle PCG} = \frac{CG}{\sin\angle GPC}$$

$$\therefore \boldsymbol{PG} = \frac{CG \cdot \sin\angle PCG}{\sin\angle GPC} \quad (27)$$

For ΔCQG,

$$\angle CGQ = 180° - \angle PGC \quad (28)$$

$$\angle GCQ = \angle C - \angle PCG \quad (29)$$

$$\angle GQC = 180° - \angle CGQ - \angle GCQ$$
$$= 180° - (180° - \angle PGC) - (\angle C - \angle PCG)$$
$$\therefore \angle GQC = \angle PGC + \angle PCG - \angle C \quad (30)$$

By Law of Sines,

$$\frac{GQ}{\sin\angle GCQ} = \frac{CG}{\sin\angle GQC}$$

$$\therefore \boldsymbol{GQ} = \frac{CG \cdot \sin\angle GCQ}{\sin\angle GQC} \quad (31)$$

From (27) and (31),

$$PG + GQ = \frac{CG \cdot \sin\angle PCG}{\sin\angle GPC} + \frac{CG \cdot \sin\angle GCQ}{\sin\angle GQC}$$

$$\therefore PQ = CG \cdot \left[\frac{\sin\angle PCG}{\sin\angle GPC} + \frac{\sin\angle GCQ}{\sin\angle GQC}\right]$$

$$\therefore PQ = CG \cdot \left[\frac{\sin\angle \mathrm{PCG}}{\sin(180° - \angle \mathrm{PCG} - \angle \mathrm{PGC})} + \frac{\sin(\angle \mathrm{C} - \angle \mathrm{PCG})}{\sin(\angle \mathrm{PGC} + \angle \mathrm{PCG} - \angle \mathrm{C})}\right]$$

Let CS=m, HG=d and $\angle \mathrm{HCG} = \theta$. Using eqns. (14) to (31),

$$\boldsymbol{PQ} = \frac{2}{3}m \cdot \left[\frac{\sin\left\{\cos^{-1}\left(\frac{b^2+m^2-\left(\frac{c}{2}\right)^2}{2\cdot b\cdot m}\right)\right\}}{\sin\left(\sin^{-1}\left(\frac{HC\cdot\sin\theta}{d}\right) - \cos^{-1}\left(\frac{b^2+m^2-\left(\frac{c}{2}\right)^2}{2\cdot b\cdot m}\right)\right)} + \frac{\sin\left(\angle C - \cos^{-1}\left(\frac{b^2+m^2-\left(\frac{c}{2}\right)^2}{2\cdot b\cdot m}\right)\right)}{\sin\left(180° - \sin^{-1}\left(\frac{HC\cdot\sin\theta}{d}\right) + \cos^{-1}\left(\frac{b^2+m^2-\left(\frac{c}{2}\right)^2}{2\cdot b\cdot m}\right) - \angle C\right)}\right]$$

This formula is derived with respect to Figure 5.
If three sides of an obtuse scalene triangle are given, using eqns. (14) to (31) and the final formula, we can find the length of portion of Euler line inside any obtuse scalene triangle.

RESULTS

If three sides of any non-right scalene triangle are given, then we can find the length of portion of Euler line that lies inside of any acute or obtuse scalene triangle using the two formulas derived in this paper.

Thus, by using these two methods and formulas, we can find the length of portion of Euler line which is in the interior of any acute or obtuse scalene triangle.

APPLICATIONS

Finding the length of portion of Euler line that lies inside of an acute or obtuse scalene triangle is very useful in solving certain complex pure geometry problems. It is also useful in the fields of engineering and construction. It also has applications in the fields of design.

FUTURE CONSIDERATIONS OF THIS PAPER

1. As the formulas are very long, so if possible, I will simplify the formulas.
2. As there are two formulas for two different types of triangles, so if possible, I will make one formula for both- acute and obtuse scalene triangle.

AKNOWLEDGEMENTS

The author wishes to thank U. M. Prajapati and U. H. Pankhania for their great help in this paper. He also wishes to thank his physics and mathematics teachers, A. B. Narang, K. G. Makhija and I. S. Chauhan for their guidance.

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