The International Young Researchers' Conference
DOI: doi.org/10.34614/iyrc0050

# The effect of orbital resonance on the stability of a planetary system

Christopher Lee
Seoul International School, South Korea
chrisjlee2002@gmail.com

Abstract

A possible idea could be that there might have been extra celestial bodies in the solar system in the distant past, but their resonance with other bodies could have caused them to be ejected out of the system or collide with another body. Orbital resonance in the domain of planetary motion happens when the orbital periods of any two celestial bodies have a simple integer ratio. The objective of this research is to further explore the effects of orbital resonances on the stability of a planetary system. To find out the effect of this resonant cycle on planetary motion, a simulation of the artificial solar system was created. The effect of resonance was measured by the standard deviation of the radius of each planet's orbit. As a result of the simulation, when the ratio of the periods of two plants formed a simple integer fraction, it was confirmed that a clear resonance can be observed. The resonance situation was also found to be significantly reduced just by shifting the resonance fraction by a small percentage. It is expected that this research will provide a better understanding of orbital resonance on the stability of a planetary system.

*Keywords: Orbital Resonance, Stability of Planetary System*

## Introduction

The stability of solar system is important and oldest problem of physics of astronomy. Understanding how the stability of our solar system is influenced is crucial for further comprehension of the physics of astronomy (Batygin et al., 2008; Ito et al., 2002; Laskar, 2013; Murray et al., 2001). Exactly how a planetary system's stability is maintained still remains a mystery among the scientific community. Understanding of the solar system revolutionized by the discovery that the orbits of the planets are inherently chaotic. In certain extreme cases, these chaotic motions can affect the relative positions of celestial bodies and can even eject one out of the system. Furthermore, the axis rotations may also alter into a chaotic state, imposing dire effects to biologically interesting planets. Recent advancements in technology and algorithms have allowed scientists in this field to simulate the solar system and further understand the idea of stability in our planetary system.

Orbital resonance can lead to instability of system(Malhotra,1994; Peale, 1976). Resonance is a phenomenon we experience in our daily lives. Pushing a friend in a swing set at its natural frequency can rapidly increase the swing's amplitude. Metronomes on a common platform synchronize over time (Pantaleones, 2002).

Likewise, orbital resonance in the domain of planetary motion happens when the orbital periods of any two celestial bodies have a simple integer ratio. In most situations, these resonances between bodies can cause gravitational influence on each other to greatly magnify, which can lead to an instability in the system, producing chaotic motions in planets, and even potentially causing bodies to be shot out into vast space.

The objective of this research is to further explore the effects of orbital resonances on the stability of a planetary system, specifically between two or among three planetary bodies, which all orbit a common star. In order to find out if this resonance period is favored in the planetary motion, an artificial solar system is created as a simulation. Using several resonance ratios and manipulating certain qualities of a celestial body, simulations are created to further determine whether a planetary system with orbital resonance is stable, or a non-resonant system is stable.

## Experimental Methods

For this orbital mechanical simulation, differential equations can calculate the future orbits of each celestial body, once their initial positions and velocities, and equation for their accelerations are given. In planetary motion, accelerations are always determined by the inverse square law.

$$a = G\frac{m}{r^2} \qquad (1)$$

Three-body problems are famously known for being unsolvable by numerous analytical methods. Thus, a differential equation solver named NDSolve, a command from Mathematica (NDSolve, n.d.), was used. Once all the initial conditions are entered, NDSolve produces a possible trajectory of a future orbit as a numerical solution.

The hypothesis used for this study is that planetary motion should be affected by resonance similar to any other periodic motion. The independent variables are integer ratio orbital period resonance, such 4/5 and continuous ratio of orbital periods. The dependent variables consist of Initial positions and velocities were calculated so that the planet's periods would have the desired integer or real number ratio above and resulting orbits. The controlled variables are that one planet was always positioned at the Earth distance from the Sun with a period of 1 Earth ear with the mass of the Earth.

The simulation is essentially a "Three-body Problem" for being unsolvable through analytical methods. Computational or numerical method are needed. Although this research is done mainly through Mathematica, other possible computational/numerical differential equations can be used to solve this problem as well. We determine the initial positions of three celestial bodies that will be used for the simulation. One of them is the Sun, which will always be located at (0, 0), and the other one is the Earth, which will be located at (1 AU, 0), where AU is the Astronomical Unit which is the distance from the center of the Earth to the center of the Sun. The third planet will be located based on the resonance ratio.

The orbital resonance refers to the resonance of periods, not distances. However, the initial conditions are specified with distances and velocities. Kepler's Law shows that the period and the distance have the following relationship:

$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{Gm}} \qquad (2)$$

Therefore, the ratio between two periods can be expressed as:

$$\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{\frac{3}{2}} \qquad (3)$$

In this research, r1 is always 1 AU. And r2 is computed by using the following formula with the resonance frequency of T2/T1:

$$r_2 = \left(\frac{T_1}{T_2}\right)^{-\frac{2}{3}} r_1 = \left(\frac{T_2}{T_1}\right)^{\frac{2}{3}} r_1 \qquad (4)$$

The initial position is determined using the distance found in Equation 3. The initial velocities are calculated to make the orbit circular. It can be obtained by equating the centripetal acceleration to the gravitational acceleration, then solving this equation for v.

$$\frac{v^2}{r} = G\frac{m}{r^2} \quad (5)$$

$$v = \sqrt{\frac{Gm}{r}} \quad (6)$$

For simplicity, all bodies are positioned on the x axis and all initial velocities are pointed to the positive y direction.

In summary, because the Sun's and Earth's positions are already determined, the only initial position to be determined is that of the third planet. It is determined based on the specified resonance ratio using Equation 4. The Sun is set to be initially stationary. Other two bodies' initial velocities were determined by Equation 6. Thus, the only independent variable is the resonance ratio.

The accelerations are continually calculated, including the initial ones, using the vector sum of Equation 1. Therefore, only the inverse square law formula is entered without any initial conditions.

We calculate T2, the period of the second body, i.e., the Earth's period, i.e., one Earth year, using Equation 2. This T2 will become the time unit in the analysis, since using units such as seconds is awkward to understand intuitively.

With these initial conditions, we run the simulation for 100,000 T2 (Earth years) using NDSolve in Mathematica, or any other numerical differential equation solver in any other languages.

## Results and Discussion

The solar system has been around for billions of years. This fact might give some assurance on its stability, but ever since Newton's time, its instability has been considered. For this research, a greatly simplified solar system only consisting of the Sun, Earth, and Mars, using their own masses and positions, for example. When the simulation was run for 100,000 Earth years, the computation abruptly goes wrong around 7,274 Earth Years as shown in Figure 1.

What causes this sudden error is not clear. One implication is that there is something about the numerical differential equation solution algorithm that is beyond my understanding. What makes it surprising is how sudden this result is. There is clearly no indication of instability until the last moment, as shown in the 3D graph (Figure 1). The third planet will be located based on the resonance ratio. The orbital resonance refers to the resonance of periods, not distances. However, the initial conditions are specified with distances and velocities. Kepler's Law is used as the relationship for the period and the distance.

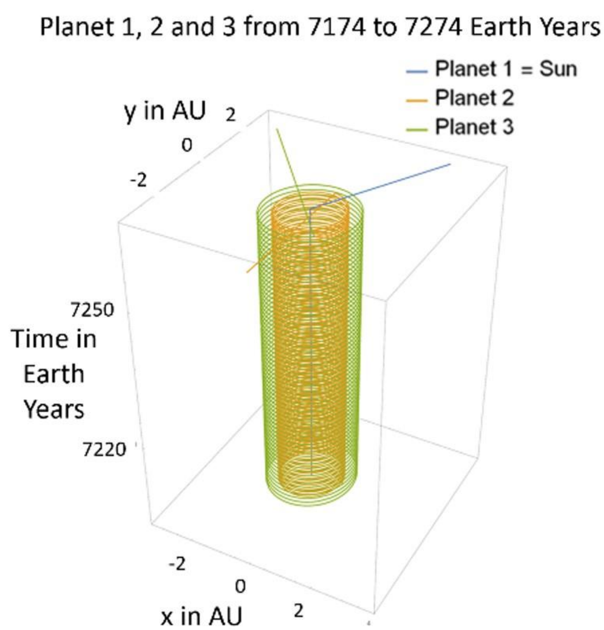


FIGURE 1. Differential equation error.

Four body resonance were examined with the following initial conditions in Table 1. The difference between resonant and non-resonant orbits were compared.

TABLE 1. Four body (3:4:5) resonance initial conditions.

| Planet | Mass | Resonance Distance | Non-resonance Distance |
|---|---|---|---|
| Sun | Sun's mass | 0 AU | 0 AU |
| Planet 2 | Earth's mass | 1 AU | 1 AU |
| Planet 3 | Earth's mass | 4/3 AU | 4.03/3 AU |
| Planet 4 | Earth's mass | 5/3 AU | 5.10/3 AU |

The resonant simulation showed noticeable variation in orbit distances, as shown in Figure 2, whereas non-resonant one was practically flat as shown in Figure 3. The simulation was set to run for 100,000 Earth years. However, it became unstable after 5,711 Earth years and ended prematurely. The outer planet was affected more severely when in that specific orbit's distance from the Sun. Nevertheless, the periods seemed to have been affected mutually. When the ratio is deviated by 1%, a significant change in the behavior of the orbit was observed. In a non-resonant system, the orbit contained chaos-free flat lines. The simulation itself lasted longer for the non-resonant orbits, lasting for around 36,651 Earth years, compared to the 5,711 Earth years of the simulation for the resonant orbits.

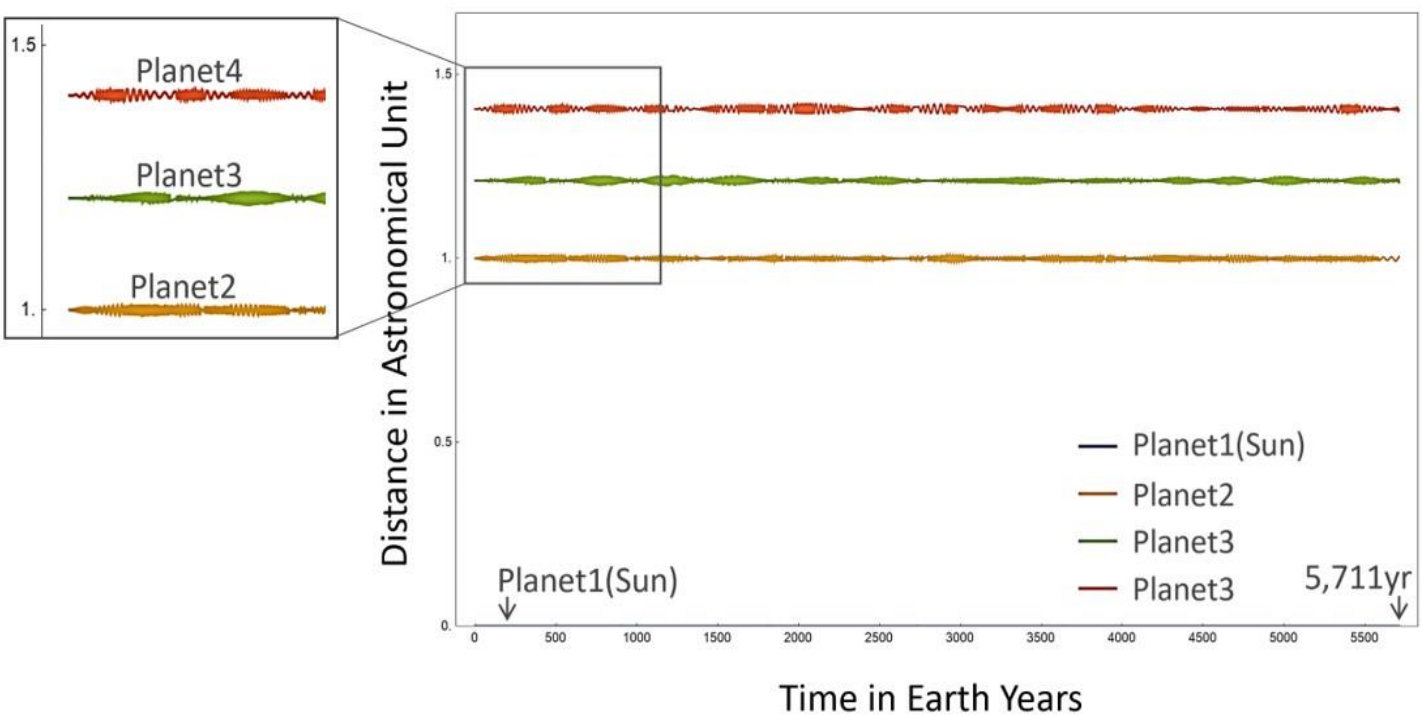


FIGURE 2. Planets distance from sun with (3:4:5) resonance.

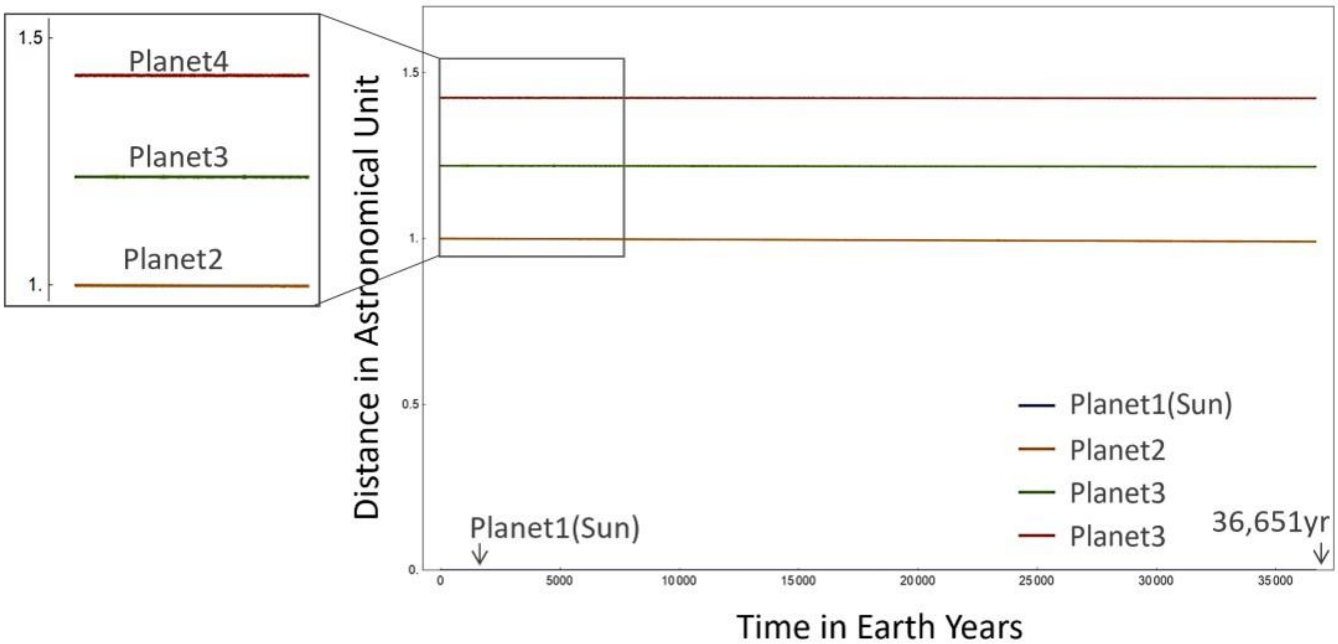


FIGURE 3. Planet distance from sun with (3:4.04:5) non-resonance.

After running several similar simulations using different ratios such as, 1:2 or 1:2:3, and observing that even a 1% deviation from the integer ratios produce dramatic changes, a more thorough simulation was done to see where exactly the resonance exists and where it is unstable. Instead of plotting all the possible orbital distances, their standard deviations were calculated and plotted in Figure 4(a).

Excluding “Planet 1”, which is the Sun at the center of the system, the three remaining planets were positions in a 1:x:3, where x varies from 1.05 to 2.95, which basically means that Planet 3 starts from right next to Planet 2, all the way to write next to Planet 4 in Figure 5. As expected, as in the beginning, the interference was strongly influenced by Planets 2 and 3. When Planet 3 was in the middle, at around the ratio of 2, both Planets 2 and 4 are equally distributed in the data.

However, as soon as Planet 3's orbit shifts closer to that of Planet 4, a similar disturbance that was observed between Planets 2 and 3 was also seen between Planets 3 and 4. Throughout the process, the disturbance expressed as a standard deviation peaks only when the orbits are in some simple integer ratios. The clear ratios had (n+1)/n on the left side of the orbit period ratio of 2, and 3n/(n+1) on the right side of 2.

One of the mysteries is the nature of the ratios when two planets are very close to each other. Integer ratios in the forms of (n+1)/n or 3n/(n+1) are visible, and at these exact points, the sudden peaks of instability exist. However, as x shifts much closer to 1, these forms of integer ratios start to fail as shown in Figure 4(b). When two planets are close, their standard deviation almost exploded. One implication is that since Planets 2 and 3 or Planets 2 and 4 are so close to each other, they must be almost in a collision course.

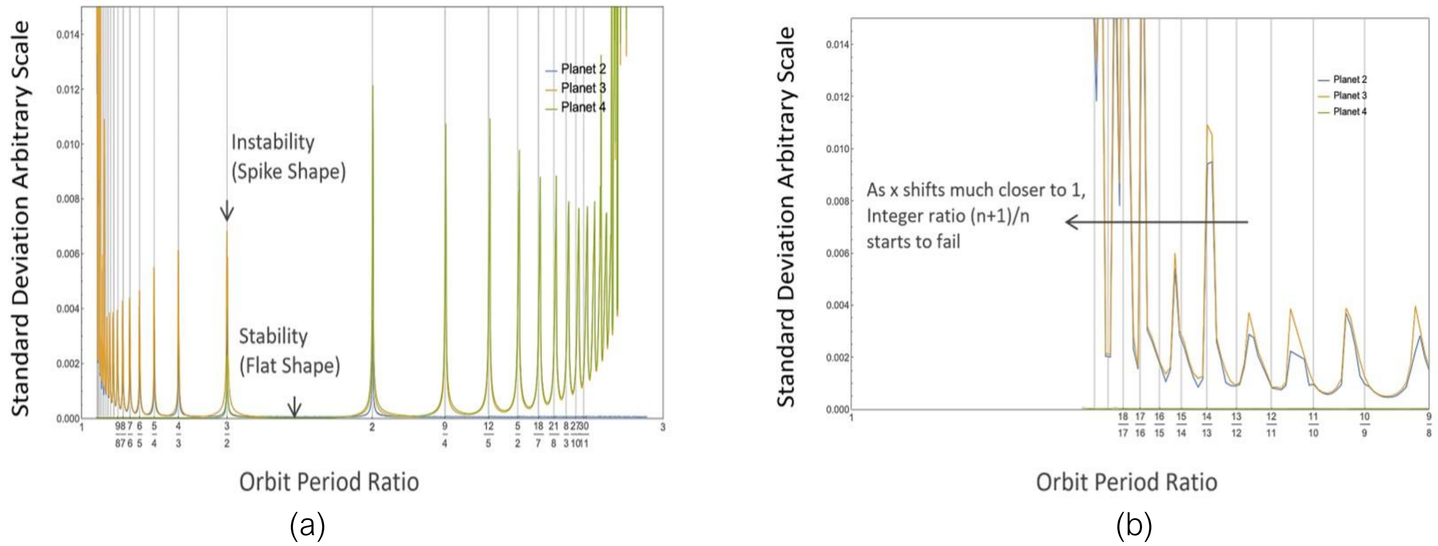


FIGURE 4. (a) 1:x:3 ratio (b) close-up view of 1:x:3 ratio.

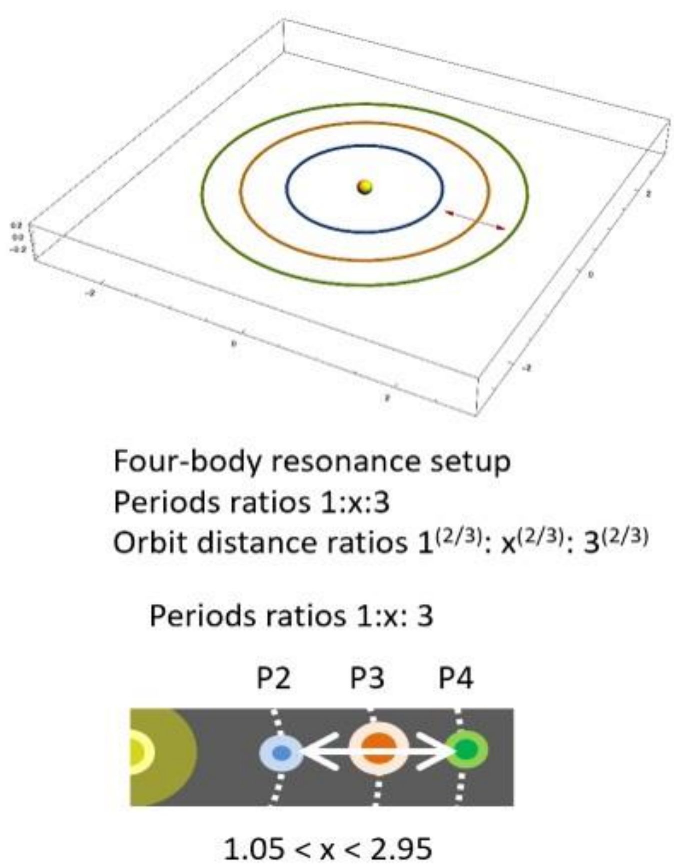


FIGURE 5. Four-body resonance setup.

Instability caused by orbital resonance may explain why such orbital resonance do not exist in our solar system. Table 2 reveals the orbital period ratio between every pair of planets in our solar system. For instance, the ratio between Jupiter and Saturn is around 0.403. Although this number appears to be extremely close to 0.4, therefore having an integer ratio of 2:5, as seen from the standard deviations and the distance graphs previously shown, even the slightest deviation of 1% could make a difference between a resonant and non-resonant planetary system. The same goes for the 0.391 ratio between Mercury and Venus, which also seems to be extremely close to 0.4, but not close enough to trigger resonant behavior.

## Conclusion

When a planetary system enters resonance, it generates a positive feedback loop, essentially gradually amplifying the planets' gravitational pulls on each other, producing instability. The simulation with various ratios of resonance all illustrates that orbital resonance leads to instability. The ratio of the periods of the planets seems extremely close to an integer ratio, yet being slight off makes them stable. A possible idea could be that there might have been extra celestial bodies in the solar system in the distant past, but their resonance with other bodies could have caused them to be ejected out of the system or collide with another body.

TABLE 2. Resonance ratios in the solar system.

| | Mercury | Venus | Earth | Mars | Jupiter | Saturn | Uranus | Neptune |
|---|---|---|---|---|---|---|---|---|
| Mercury | 1 | 2.554 | 4.152 | 7.809 | 49.254 | 122.267 | 348.840 | 684.217 |
| Venus | 0.391 | 1 | 1.625 | 3.057 | 19.283 | 47.867 | 136.569 | 267.868 |
| Earth | 0.241 | 0.615 | 1 | 1.881 | 11.862 | 29.447 | 84.015 | 164.789 |
| Mars | 0.128 | 0.327 | 0.532 | 1 | 6.307 | 15.657 | 44.670 | 87.615 |
| Jupiter | 0.020 | 0.052 | 0.084 | 0.159 | 1 | 2.482 | 7.082 | 13.892 |
| Saturn | 0.008 | 0.021 | 0.034 | 0.064 | 0.403 | 1 | 2.853 | 5.596 |
| Uranus | 0.003 | 0.007 | 0.012 | 0.022 | 0.141 | 0.350 | 1 | 1.961 |
| Neptune | 0.001 | 0.004 | 0.006 | 0.011 | 0.072 | 0.179 | 0.510 | 1 |

One idea for future research is to analyze the orbital resonances of other planets in the solar system, such as Saturn or Neptune, and also find out the effects of orbital resonance of celestial bodies on smaller bodies such as asteroids, specifically, analyze the Kirkwood gaps. Another is to find out what might have exactly happened to the integer resonance ratios in Figure 4.

## References


Batygin, K., & Laughlin, G. (2008). On the dynamical stability of the solar system. The Astrophysical Journal, 683(2), 1207.

Ito, T., & Tanikawa, K. (2002). Long-term integrations and stability of planetary orbits in our Solar system. Monthly Notices of the Royal Astronomical Society, 336(2), 483-500.

Laskar, J. (2013). Is the solar system stable?. In Chaos (pp. 239-270). Birkhäuser, Basel.

Malhotra, R. (1994). Nonlinear resonances in the solar system. Physica D: Nonlinear Phenomena, 77(1-3), 289-304.

Murray, N., & Holman, M. (2001). The role of chaotic resonances in the solar system. *Nature*, 410, 773-779.

NDSolve. (n.d.). Retrieved from https://reference.wolfram.com/language/ref/NDSolve.html

Pantaleone, J. (2002). Synchronization of metronomes. *American Journal of Physics*,70, 992–1000.

Peale, S. J. (1976). Orbital resonances in the solar system. Annual review of astronomy and astrophysics, 14(1), 215-246.